Passive scalar advected by a very irregular random velocity field

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It is shown that the anomalous exponent of a scalar advected by an extremely irregular turbulent field found by Gawedzki and Kupiainen [Phys. Rev. Lett. **21**, 3834 (1995)] in a special limit can be derived also in a more general case with the velocity viscous scale kept finite. [S1063-651X(97)06507-0]

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I consider the problem of a passive scalar field advected by a random velocity field. The advection is governed by the equation

$$(\partial_t + v^\alpha \nabla^\alpha - \kappa\Delta)\theta = \phi. \qquad (1)$$

The Laplacian term corresponds to the molecular diffusion with diffusivity κ. Let us assume that the scalar field is randomly pumped into the system by a pumping force ϕ. The statistics of the pumping is Gaussian with the pair-correlation function $\langle \phi(r,t)\phi(0,0)\rangle = \delta(t)\chi(r)$, where $\chi(r)$ is decaying on some scale L. The turbulent field v is assumed to be δ-correlated Gaussian with the velocity pair-correlation function

$$\langle v^\alpha(0,0) v^\beta(t,r)\rangle = \delta(t)[\langle v^2\rangle \delta^{\alpha\beta} - \mathcal{K}^{\alpha\beta}(r)]. \qquad (2)$$

Here $\mathcal{K}^{\alpha\beta}(r)$ is an eddy diffusivity tensor usually defined as [1]

$$\mathcal{K}^{\alpha\beta} = \frac{D}{r^\gamma}(r^2\delta^{\alpha\beta} - r^\alpha r^\beta) + \frac{D(d-1)}{2-\gamma}\delta^{\alpha\beta} r^{2-\gamma}. \qquad (3)$$

We see that when the velocity field becomes extremely irregular (the scaling exponent γ tends to 2) the eddy diffusivity tensor has a singularity. One way to handle this was used in [2] by considering the special limit when $D \propto (2-\gamma)$, i.e., decreasing the magnitude of the velocity field together with the degree of smoothness. Since the main point of interest in [2] was the (anomalous) exponent ζ_4 of the fourth-order correlation function, it is natural to assume that the dimensionless ζ_4 cannot depend on the definition of the dimensional quantity D. Indeed, it will be shown below that the problem can be consistently considered while all physical quantities are kept finite and the result for ζ_4 coincides with that derived by Gawedzki and Kupiainen.

To treat the singularity accurately we regularize the eddy diffusivity tensor in the ultraviolet limit by introducing the cutoff at some viscous scale r_η as it has been suggested in [4]. Physically, this means that the velocity field is smoothed by viscosity at scales less than r_η and, as a result, the velocity pair-correlation function should be quadratic at such small scales. One may use, for instance, the generalization

$$\mathcal{K}^{\alpha\beta} = \frac{\widetilde{D}(r/r_\eta)^2}{(1+r^2/r_\eta^2)^{\gamma/2}}\left(\delta^{\alpha\beta} - \frac{r^\alpha r^\beta}{r^2}\right) + \frac{\widetilde{D}(d-1)}{2-\gamma} \times \delta^{\alpha\beta}[(1+r^2/r_\eta^2)^{(2-\gamma)/2} - 1], \qquad (4)$$

where $\widetilde{D} \equiv D r_\eta^{2-\gamma}$. As $\gamma \to 2$, Eq. (4) has the finite limit

$$\mathcal{K}^{\alpha\beta} \to D\left(\delta^{\alpha\beta} - \frac{r^\alpha r^\beta}{r^2}\right) + \frac{D(d-1)}{2}\ln[1+(r/r_\eta)^2]\delta^{\alpha\beta}.$$

We want to study the properties of the correlation functions of a passive scalar field inside the convective interval. The upper limit of this interval is determined by the pumping scale L. To find the general expression for the diffusion scale one has to compare $\mathcal{K}^{\alpha\beta}$ and κ, which gives

$$r_d = \sqrt{\left(\frac{\kappa(2-\gamma)}{D} + r_\eta^{2-\gamma}\right)^{2/(2-\gamma)} - r_\eta^2}. \qquad (5)$$

If $2-\gamma$ is the smallest parameter in the problem, then

$$r_d = r_\eta\left[\exp\left(\frac{2\kappa}{(d-1)D}\right) - 1\right]^{1/2}. \qquad (6)$$

If the condition $\kappa < D r_\eta^{2-\gamma}$ is satisfied then the diffusion scale is less than the viscous scale $r_d < r_\eta$ and the convective interval is $r_\eta \ll r \ll L$. I shall show that, as long as $\ln(r/r_\eta) \gg 1$, the statistics of the scalar is close to Gaussian in the convective interval at $\gamma = 2$, as it was sugested by Falkovich [3]. Considering then the perturbation theory with respect to the small parameter $2-\gamma$, we shall prove that the anomalous scaling exponent was correctly calculated in [2]. The equation for the two-point correlation function $\langle \theta(0,t)\theta(r,y)\rangle = f(r)$ is

$$\hat{\mathcal{L}} f(r) = -\chi(r). \qquad (7)$$

Let us choose the pumping to be close to a steplike function

$$\chi = \begin{cases} 1, & r < L \\ 0, & r > L. \end{cases} \qquad (8)$$

The turbulent diffusion operator for an n-point object is

$$\hat{\mathcal{L}} = \sum_{i,j} K^{\alpha\beta}(r_{ij}) \nabla_i^\alpha \nabla_j^\beta, \qquad (9)$$

where $\boldsymbol{r}_{ij}=\boldsymbol{r}_i-\boldsymbol{r}_j$ The solution of Eq. (7) inside the convective interval in the main logarithmic order is

$$f(0)-f(r)=\frac{r^2}{\ln\left(\frac{r}{r_\eta}\right)}, \quad r_\eta \ll r \ll L. \tag{10}$$

We see that the pair-correlation function is almost ''diffusive'' in the convective interval. At $r>L$, it decays by the law $f(r)\propto r^{2-d}$ [4]. We thus restrict ourselves by $d>2$; the same restriction has been assumed in [2]. The fourth-order correlation function satisfies the equation

$$\hat{\mathcal{L}}F=-\sum \chi(r_{ij})f(r_{kl}). \tag{11}$$

Since the operator of turbulent diffusion is similar to the usual diffusion operator in this case, we guess that the answer at $\gamma=2$ is close to Gaussian $F_g=f(r_{12})f(r_{34})+f(r_{13})f(r_{24})+f(r_{14})f(r_{23})$. To show that this is the case, we develop a perturbation theory, dividing the operator $\hat{\mathcal{L}}$ into four pieces

$$\hat{\mathcal{L}}=\hat{\mathcal{L}}_0+\hat{\mathcal{L}}_1+\hat{\mathcal{L}}_2+\hat{\mathcal{L}}_3, \tag{12}$$

$$\hat{\mathcal{L}}_0=D\delta^{\alpha\beta}(d-1)\ln\left(\frac{r}{r_\eta}\right)\nabla_i^\alpha\nabla_j^\beta, \tag{13}$$

$$\hat{\mathcal{L}}_1=D\delta^{\alpha\beta}(d-1)\ln\left(\frac{r_{ij}}{r}\right)\nabla_i^\alpha\nabla_j^\beta, \tag{14}$$

$$\hat{\mathcal{L}}_2=D\left(\delta^{\alpha\beta}-\frac{r^\alpha r^\beta}{r^2}\right)\nabla_i^\alpha\nabla_j^\beta, \tag{15}$$

$$\hat{\mathcal{L}}_3=-\frac{(2-\gamma)}{2}\ln\left(\frac{r_{ij}}{r_\eta}\right)\left(\delta^{\alpha\beta}-\frac{r^\alpha r^\beta}{r^2}\right)\nabla_i^\alpha\nabla_j^\beta. \tag{16}$$

Here $r^2=\Sigma_{i,j}r_{ij}^2$. Our first goal is to verify that the Gaussian solution that we proposed is correct at $\gamma=2$, that is, the corrections due to operators $\hat{\mathcal{L}}_1$ and $\hat{\mathcal{L}}_2$ are small in $\ln^{-1}(r/r_\eta)$ in comparison to the unperturbed solution. After that, following [2], one may start the perturbation theory with respect to $\hat{\mathcal{L}}_3$.

Let us denote by F^i the correction due to the operator $\hat{\mathcal{L}}_i$,

$$\hat{\mathcal{L}}_0\delta F^i=\hat{\mathcal{L}}_iF_g. \tag{17}$$

The total correction is $\delta F=\delta F^1+\delta F^2+\delta F^3$. In order to develop such a theory one should solve Eq. (17). Since we are especially interested in the irredicible part of the fourth-order correlation function it is convenient to develop the theory in the coordinates that use the translational invariance and diagonalize the operator $\hat{\mathcal{L}}_0$. The following choice of the variables satisfies this condition [2]:

$$\boldsymbol{r}_{12}=\boldsymbol{x}, \quad \boldsymbol{r}_{34}=\boldsymbol{z}, \quad \boldsymbol{r}_{23}=\frac{1}{\sqrt{2}}\boldsymbol{y}-\frac{1}{2}\boldsymbol{x}-\frac{1}{2}\boldsymbol{z}. \tag{18}$$

One may choose other permutations as well. In these variables, for the three-dimensional problem, the operator $\hat{\mathcal{L}}_0$ is

$$\hat{\mathcal{L}}_0=\ln\left(\frac{r}{r_\eta}\right)[\Delta_x+\Delta_y+\Delta_z]. \tag{19}$$

The Green's function $\mathcal{G}$ of $\hat{\mathcal{L}}_0$ is

$$\mathcal{G}=\ln^{-1}\left(\frac{r}{r_\eta}\right)\frac{1}{[(\boldsymbol{x}-\boldsymbol{x}')^2+(\boldsymbol{y}-\boldsymbol{y}')^2+(\boldsymbol{z}-\boldsymbol{z}')^2]^{7/2}}.$$

From Eq. (17) we obtain

$$\hat{\mathcal{L}}_0\delta F^1=\frac{3r_{ij}^2}{\ln\left(\frac{r_{ij}}{r_\eta}\right)\ln\left(\frac{r_{kl}}{r_\eta}\right)}+\frac{4\boldsymbol{r}_{ij}\cdot\boldsymbol{r}_{kl}}{\ln\left(\frac{r_{ij}}{r_\eta}\right)\ln\left(\frac{r_{kl}}{r_\eta}\right)}\ln\left(\frac{r_{ik}r_{jl}}{r_{il}r_{jk}}\right).$$

The structure of the correction is naturally divided into two parts. As may be seen directly from Eq. (20), the terms of the type $r_{ij}^2\ln^{-1}(r_{ij}/r_\eta)\ln^{-1}(r_{kl}/r_\eta)$ produce the correction $r_{12}^4\ln^{-2}(r_{12}/r_\eta)\ln^{-1}(r_{34}/r_\eta)$, which is small in $\ln^{-1}(r/r_\eta)$ compared to the reducible Gaussian part. To calculate the correction that comes from the remaining term, a more delicate analysis should be done. This correction may be represented as an integral

$$\sum_{i\neq j\neq k\neq l}\int\frac{\boldsymbol{r}_{ij}\cdot\boldsymbol{r}_{kl}}{\ln\left(\frac{r_{ij}}{r_\eta}\right)\ln\left(\frac{r_{kl}}{r_\eta}\right)}\ln\left(\frac{r_{ik}r_{jl}}{r_{il}r_{jk}}\right)\mathcal{G}(r,r')dr'. \tag{20}$$

This integral is regular at the low limit, but diverges at the upper limit. We are concerned with the divergent parts of Eq. (20). The integration may be divided into two parts $r_\eta\leqslant r'\leqslant r$ and $r\leqslant r'\leqslant L$. The integral (20) cannot be calculated explicitly; however, some definite estimations can be made.

Let us start to expand the integrand in r over L and try to calculate each such term. Since we use the pair-correlation function in the main logarithmic order this integral will produce the term that diverges as L^4 and L^2. The terms with $n=3$ and $n=1$ are forbidden by an isotropy of the problem. We pass to the Gawedzky-Kupiainen variable, choosing in each term this coordinate as x and z,

$$\int_0^L\frac{d^3x'd^3y'd^3z'\boldsymbol{x}'\cdot\boldsymbol{z}'\ln\left(\frac{|\sqrt{2}y+x-z||\sqrt{2}y-x+z|}{|\sqrt{2}y+x+z||\sqrt{2}y-x-z|}\right)}{\ln\left(\frac{r}{r_\eta}\right)\ln\left(\frac{x'}{r_\eta}\right)\ln\left(\frac{z'}{r_\eta}\right)[(\boldsymbol{x}-\boldsymbol{x}')^2+(\boldsymbol{y}-\boldsymbol{y}')^2+(\boldsymbol{z}-\boldsymbol{z}')^2]^{7/2}}.$$

To prove that the perturbation theory is valide we should show that correction to zero mode is zero. There are two different regions of integration that may contribute to the correction. In the first region two integrals are determined inside the convective interval and a third is on a scale much larger than convective interval

$$\int_0^L d^3y' \frac{\boldsymbol{x}\cdot\boldsymbol{z}}{\ln\left(\frac{x}{r_\eta}\right)\ln\left(\frac{z}{r_\eta}\right)} \ln\left(\frac{1+\frac{(x-z)^2}{2y'^2}-\frac{3}{2}\left(\frac{y'(x-z)}{y'^2}\right)^2}{1+\frac{(x+z)^2}{2y'^2}-\frac{3}{2}\left(\frac{y'(x+z)}{y'^2}\right)^2} \right)$$

$$=\int_0^L d^3y' \frac{2\boldsymbol{x}\cdot\boldsymbol{z}}{\ln\left(\frac{x}{r_\eta}\right)\ln\left(\frac{z}{r_\eta}\right)}\left(-\frac{\boldsymbol{x}\cdot\boldsymbol{z}}{y^2}+3\frac{(\boldsymbol{x}\cdot\boldsymbol{y}')(\boldsymbol{z}\cdot\boldsymbol{y}')}{y'^4}\right)=0.$$

We see that after integration over angles this term vanishes. It is yet an unsolved problem to show analytically that the contributions from all other regions of integration are vanishing. However, since the coefficient before the logarithmically divergent part does not contain any parameters it may be calculated numerically. Using the Monte Carlo method, we verify that this integral is indeed zero. We have proved that the correction F^1 is small in $\ln^{-1}(r/r_\eta)$ with respect to the Gaussian part of a solution.

Finally, we show that the last part of the operator, namely, $r_{ij}^\alpha r_{ij}^\beta r_{ij}^{-2}\nabla_i^\alpha\nabla_j^\beta$, does not affect the answer. Since the cross term operator $\hat{\mathcal{L}}_2$ has a zero scaling then the correction F^2 obeys the equation

$$\hat{\mathcal{L}}_0\delta F^2=r_{ij}^2\ln^{-2}\left(\frac{r}{r_\eta}\right). \tag{21}$$

Therefore, $\delta F^2\sim f^2(r)\ln^{-1}(r/r_\eta)$. As can be seen, the perturbation theory in $\ln^{-1}(r/r_\eta)$ is regular and the solution for $\gamma=2$ is Gaussian in $\ln^{-1}(r/r_\eta)$. Note that the near-Gaussian solution found at $r\gg r_\eta$ can be matched naturally with the near-Gaussian solution in the small-scale viscous-convective interval $r_d\ll r\ll r_\eta$ (which exists if $\kappa\ll\widetilde{D}$), where $\gamma=0$ and $f(r)\sim r_\eta^2\ln(r_\eta/r)$, etc. [5].

Therefore, the perturbation theory for $2-\gamma$ was started in [2] with the correct zero approximation. The next step is to develop the perturbation theory for $2-\gamma$, keeping the largest term in $\ln^{-1}(r/r_\eta)$. Since the operator $\hat{\mathcal{L}}_3$ has the same logarithmic dependence as $\hat{\mathcal{L}}_0$, the respective logarithms cancel each other. At any step of the $2-\gamma$ expansion the answer in the main logarithmic order is the same as when we substitute a constant factor instead of the logarithm (as has been done in [2]). That means that the anomalous scaling exponent was calculated correctly in [2]. One may see the qualitative difference between the small-scale behavior of two Gaussian limits $\gamma=0$ and $\gamma=2$: in the first case, non-Gaussianity decreases as the scale decreases [5], while in the second case it is the other way around.

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